Hopf Monoids of Ordered Simplicial Complexes

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Coauthors

This is joint work with Jeremy Martin, from University of Kansas, and José Samper, from Pontificia Universidad Católica de Chile.

Figure: Jeremy Martin

Figure: José Samper

Simplicial Complexes: Glossary

A **simplicial complex** Γ on a set I is a nonempty subset of 2^I closed under inclusion. Elements of Γ are called faces. It is enough to specify the largest (under inclusion) faces. Its **rank** is the size of the largest subset in Γ. The **dimension** is rank minus one. It is **pure** if all maximal (under inclusion) sets have the same size.

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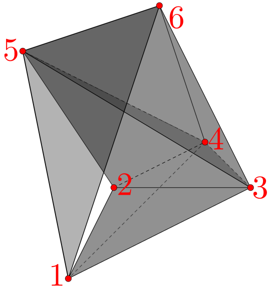
- The **restriction** of Γ to $J \subset I$ consist on the subsets in Γ completely contained in J.
- The **link** of a face $\gamma \in \Gamma$ consist of $\{\tau \in \Gamma : \gamma \cup \tau \in \Gamma, \gamma \cap \tau = \varnothing\}$. The link is a simplicial complex with grounded in the **complement** of γ.

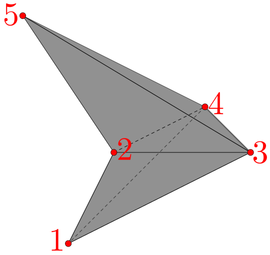
Simplicial Complexes: Example

Let $\Gamma = \langle 123, 124, 134, 235, 245, 345, 156, 346, 356, 456 \rangle$.



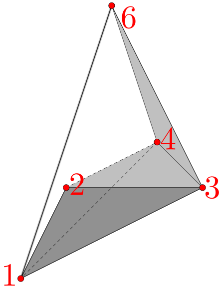
Simplicial Complexes: Example

Restriction to $\{1,2,3,4,5\}$, $\Gamma|_{12345} = \langle 123, 124, 134, 235, 245, 345 \rangle$.



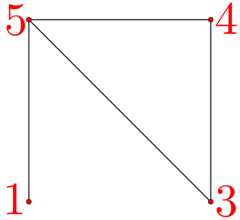
Simplicial Complexes: Example

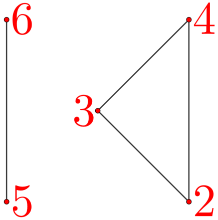
Restriction to $\{1,2,3,4,6\}$, $\Gamma|_{12346} = \langle 123, 124, 134, 16, 346\rangle$.



Simplicial Complexes: Example

Link of the vertices 6 and 1.

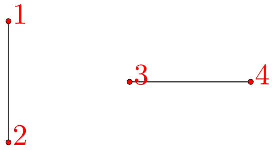


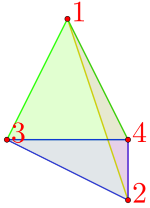


Join of simplicial complexes

If Γ_1, Γ_2 are simplicial complexes on vertex sets I_1, I_2 then the **join** $\Gamma_1 * \Gamma_2$ is the complex on $I_1 \sqcup I_2$ defined by

$$\Gamma_1 * \Gamma_2 = \{\gamma_1 \cup \gamma_2 : \ \gamma_1 \in \Gamma_1, \ \gamma_2 \in \Gamma_2\}.$$





Hopf Classes

An **ordered complex** is a pair (w, Γ) where w is any linear order on the ground set of Γ.

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An **ordered complex** is a pair (w, Γ) where w is any linear order on the ground set of Γ. We say that a **pure** ordered complex is **prefix-pure** if all its initial restrictions (hence all its initial contractions) are pure.

Main Definition

A class $\mathbf{H}$ of prefix-pure ordered complexes is called a **Hopf class** if it satisfies the following three conditions.

1. **Closure under ordered join**
2. **Closure under initial restriction**
3. **Closure under initial contraction**: If $(w, \Gamma) \in \mathbf{H}$, A is an initial segment of w and γ is the lex minimal facet of $\Gamma|_A$, then $(w', \mathrm{link}_\gamma(\Gamma)) \in \mathbf{H}$. Here w' is restricted to the complement of A.

Example: Ordered matroids

One definition of matroid

A simplicial complex is the independence complex of a matroid if and only if the restriction to any subset is pure.

Let **OMAT** be the class of all independence complexes of matroids, together with **any** linear order on the ground set. Then **OMAT** is a Hopf class.

Example: Shifted complexes

Definition

An ordered simplicial complex (w, Γ) is **shifted** if for any face $\gamma \in \Gamma$ and any vertex $e \in \gamma$ then $\gamma \cup f \backslash e$ is a face for every $f <_w e$.

Example

Example The complex $\Gamma = \langle 123, 124, \mathbf{125}, 134, \mathbf{234} \rangle$ is shifted, generated by the two highlighted facets.

Pure shifted complexes do not form a Hopf class because they are not closed under ordered join. On the other hand, the class of ordered joins of shifted complexes is a Hopf class **SHIFT**.

Example: Matroid truncation

Let (w, Γ, I) be a pure ordered complex. The **Gale order** on facets of Γ is defined as follows. Let $F = \{f_1 <_w \cdots <_w f_r\}$ and $G = \{g_1 <_w \cdots <_w g_r\}$ be two facets. Then $F \leqslant_g G$ if $f_i \leqslant_w g_i$ for every i.

Definition

Let $\mathcal{J}$ be an order ideal of this poset; then the **Gale truncation** of (w, Γ) at $\mathcal{J}$ is $(w, \Gamma_{\mathcal{J}})$, where $\Gamma_{\mathcal{J}}$ is the (pure) subcomplex of Γ generated by the facets in $\mathcal{J}$.

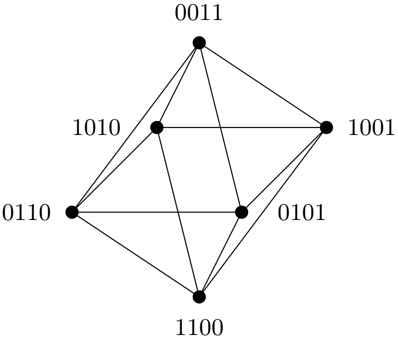
Gale truncation generalizes the notion of shiftedness: a shifted complex is just a Gale truncation of an ordered **uniform matroid**. Gale truncations of matroids *generate* a Hopf Class.

Example: Unbounded matroids

Theorem (Gelfand-Goresky-MacPherson-Serganova)

A pure complex is the independent complex of a matroid if and only if its **indicator polytope** has all edges parallel to roots of type A.

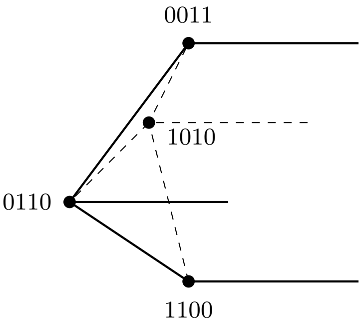
For example consider the matroid with bases $\{12, 13, 14, 23, 24, 25, 34\}$.



Example: Unbounded matroids

Definition

An unbounded matroid is the **indicator complex** of a 0/1 poly**hedron** whose edges are parallel to roots of type A.



Unbounded matroids form a hopf class $\mathbf{OMAT}_+$.

Hopf Monoids

Intuition

Families of **labeled** combinatorial structures with rules for **merging** and **breaking**.

A Hopf Monoid $(\mathbf{H}, \mu, \Delta)$ consists of

- For each finite set I a vector space $\mathbf{H}[I]$.
- A **product** $\mu_{S,T} : \mathbf{H}[S] \otimes \mathbf{H}[S] \mapsto \mathbf{H}[S \sqcup T]$.
- A **coproduct** $\Delta_{S,T} : \mathbf{H}[S \sqcup T] \mapsto \mathbf{H}[S] \otimes \mathbf{H}[S]$.

Satisfying compatibility axioms.

Main Construction

Hopf Monoid on ordered simplicial complexes

Given a Hopf class we define **product** and **coproducts** on basis elements by join and restriction/contraction respectively:

$$\mu_{I,J}((w_1,\Gamma_1)\otimes(w_2,\Gamma_2)) = \sum_{w\in\mathsf{Shuffle}(w_1,w_2)} (w_1,\Gamma_1) *_w (w_2,\Gamma_2),$$

$$\Delta_{I,J}(w,\Gamma) = \begin{cases} (w|I,\Gamma|I)\otimes(w/I,\Gamma/I) & \text{if } I \text{ is an initial segment of } w,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem

For every Hopf class $\mathbf{H}$*, the tuple* $(\mathbf{H},\mu,u,\Delta,\epsilon)$ *is a connected Hopf monoid.*

Antipode

Takeuchi Formula

The **antipode** of a Hopf monoid is given by

$$\mathsf{s}_I(h) = \sum_{\substack{I=A_1\sqcup\cdots\sqcup A_k\\ k\geqslant 1}} (-1)^k \mu_{A_1,\cdots,A_k} \circ \Delta_{A_1,\cdots,A_k}(h)$$

In some sense, this plays the role of the **inverse** map in a Hopf monoid.

Theorem

A multiplicty-free, cancellation-free, formula for the antipode in the cases of **ordered matroids** and **shifted complexes**.

We conjecture that antipode for any Hopf class is multiplicty-free.

How to compute the antipode?

The main idea comes from **Ardila-Aguiar**: consider the matroid polytope and interpret that massive cancellation as an **Euler characteristic**.

Devil is in the details

When working with **generalized permutohedra** one is lead to **braid combinatorics**: every cone is a union of braid cones, which in turn can be describe with a poset.

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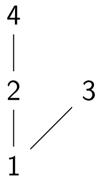
For example,

Devil is in the details

When working with **generalized permutohedra** one is lead to **braid combinatorics**: every cone is a union of braid cones, which in turn can be describe with a poset.
For example,the poset below defines a cone
$\{\mathbf{x} \in \mathbb{R}^4 : x_1 \leqslant x_2, x_1 \leqslant x_3, x_2 \leqslant x_4\}$ and it is the union of three braid cones

$$\sigma_{1234} = \{\mathbf{x} \in \mathbb{R}^4 : x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4\},$$
$$\sigma_{1243} = \{\mathbf{x} \in \mathbb{R}^4 : x_1 \leqslant x_2 \leqslant x_4 \leqslant x_3\},$$
$$\sigma_{1324} = \{\mathbf{x} \in \mathbb{R}^4 : x_1 \leqslant x_3 \leqslant x_2 \leqslant x_4\}.$$

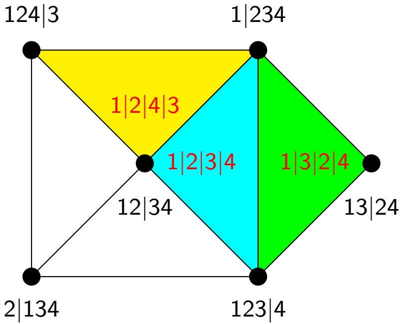


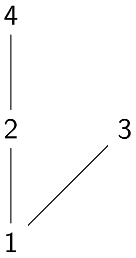
Poset cones

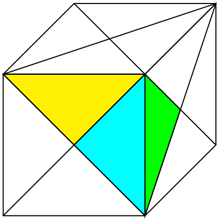
For every poset on we can associate a cone $\mathcal{C}_P$, which is a union of braid cones.

Where problems arise

The thorny issue is how to compute $\partial\mathcal{C}_P \cap \partial\mathcal{C}_Q$ for arbitrary posets P and Q.







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Maybe the real treasure was the friends we made along the way.

Thank You!